Fourier Pricing of European Options

Jacques Burrus

Burrus Financial Intelligence, Las Condes, Chile
E-Mail: Jacques.Burrus@bfi.cl
Phone: +56-99-533-4347
Address: Andacollo 1599, Providencia, Santiago, Chile

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Abstract

Fourier analysis sometimes cannot be easily applied to very common financial contracts -such as e.g. European vanilla call options - because the Fourier transform of their payoff has a singularity. This article addresses this concern by approximating the original payoff with a symmetrized trigonometric series. This approach presents the following advantages:

1. eliminate the Gibbs phenomenon for continuous payoffs,
2. convert a continuous Fourier integral into a discrete sum,
3. apply the Fast Fourier Transform algorithm for efficiency.

One can also obtain a price vector when the model is arithmetic. This methodology is benchmarked with the Black-Scholes model.

1 Introduction

Fourier transforms are a powerful tool for the valuation of financial derivatives because they provide analytical expressions for advanced models. Unfortunately, this technique also suffers from numerical instabilities that this paper seeks to address.

Section 2 derives the fundamental discounting formula for a European contract with free boundaries given an initial market condition. This formula remarkably separates the model from the payoff description. Next, section

3 explains how to approximate an arbitrary payoff profile in order to avoid damaging oscillations. Finally, section 4 shows how to efficiently compute the present value on a vector of initial conditions provided that the state variables follow an arithmetic process. This algorithm allows to design generic pricing engines based on Fourier technology that can for instance value Bermudan options. Finally, section 5 illustrates the presented methodologies with the Black-Scholes model and compares their respective computation times.

2 Fundamental Pricing Formula

A European contract makes a sole payment V at maturity T. The market also trades another asset B that has no intermediary payments.

We assume that both the payoff V of the contract and the value B of the numeraire depend on a state vector x of length D and that their ratio[1] $Y = V/B$ is square integrable. The Fourier transform[2] ψ_Y of $Y(x,T)$ can henceforth be defined as: $\psi_Y(\omega,T) = 1/(2\pi)^D \int_{x\in\mathbb{R}^D} Y(x,T)\, e^{-i\omega x} dx$.

The distribution of the state vector x is known at any future date t conditional on its initial value X_0 and its probability density function (PDF) in the equivalent martingale measure (EMM) with numeraire B is: $p^B_{X|X_0}(x,t)$. Alternatively, one can describe this distribution with the characteristic function[3] (CF): $\psi^B_{X|X_0}(\omega,t) = E^B_0\left[e^{i\omega x_t}\right] = \int_{x\in\mathbb{R}^D} p^B_{X|X_0}(x,t)\, e^{i\omega x} dx$, which is sometimes more analytically tractable than the PDF.

The fundamental pricing relation states that Y is a martingale under the EMM with numeraire B:

$$Y(X_0,0) = E^B_0\left[Y(x,T)\right]$$

Expanding this expression and replacing Y by its Fourier transform ψ_Y yields: $Y(X_0,0) = \int_{x\in\mathbb{R}^D} p^B_{X|X_0}(x,t)\left[\int_{\omega\in\mathbb{R}^D} \psi_Y(\omega,T)\, e^{i\omega x} d\omega\right] dx$. Next, inverting the integrals and recognizing the CF $\psi^B_{X|X_0}$ of the state vector x at time T leads to the pricing formula at X_0:

$$Y(X_0,0) = \int_{\omega\in\mathbb{R}^D} \psi^B_{X|X_0}(\omega,T)\, \psi_Y(\omega,T)\, d\omega$$

This expression possesses at least two desirable properties. First, there is a clear functional separation between the payoff ψ_Y and the model specification ψ^B_X, which offers modularity as one can set up distinct catalogs for

[1] The values of the payoff V and the numeraire B are usually denominated in currency units. However, any other traded asset with no intermediary payments -e.g. a Money Market Account- such as B may equally well serve as a numeraire. Therefore, the ratio $Y = V/B$ denotes the value of V in units of the asset B.

[2] The inverse Fourier transform of ψ_Y reads: $Y(x,T) = \int_{\omega\in\mathbb{R}^D} \psi_Y(\omega,T)\, e^{i\omega x} d\omega$. The Fourier transform function ψ of a real-valued function satisfies: $\psi(-\omega) = \bar{\psi}(\omega)$.

[3] The CF is the inverse Fourier transform of the real-valued PDF so that: $\psi^B_{X|X_0}(-\omega) = \bar{\psi}^B_{X|X_0}(\omega)$. Also note that: $\psi^B_{X|X_0}(0) = 1$.

payoffs and models. Secondly, discounting involves only the multiplication of the functions $\psi^B_{X|X_0}$ and ψ_Y, instead of e.g. the resolution of a linear system.

For the sake of clarity, the next sections only consider a dimension: $D = 1$. The general case follows naturally.

3 Pricing for One Initial Value

3.1 Fourier Transform of a Real Trigonometric Series

As mentioned before, some payoffs Y do not admit a Fourier transform[4].

We circumvent this difficulty by approximating the original payoff Y as a trigonometric series[5]: $\widetilde{Y}(x,T) = \alpha_0/2 + \sum_{j=1}^{M-1} \alpha_j \cos\omega_j x + \sum_{j=1}^{M-1} \beta_j \sin\omega_j x$, which is equivalent to: $\widetilde{Y}(x,T) = \sum_{j=-(M-1)}^{M-1} \psi_j e^{i\omega_j x}$, where for $j \geq 0$: $\psi_j = (\alpha_j - i\beta_j)/2$, $\psi_{-j} = \bar{\psi}_j$, and: $\omega_{-j} = -\omega_j$.

Thus, the Fourier transform ψ_Y of the proxy $\widetilde{Y}$ is zero everywhere except at the frequencies ω_j where: $\psi_Y(\omega_j, T) = \psi_j$. The previous pricing formula applied to this trigonometric series becomes a finite sum[6]: $Y(X_0, 0) = \sum_{j=-(M-1)}^{M-1} \psi^B_{X|X_0}(\omega_j, T)\psi_j$. As seen later, this approximation does not appear to significantly damage the pricing accuracy.

3.2 Determination of the Payoff Proxy

We must now determine the coefficients ψ_j of the trigonometric proxy $\widetilde{Y}$, i.e. the $N = 2M - 1$ parameters α and β. We force the proxy to match the original payoff Y at N points x_k. This leads to the linear system of N equations and N variables: $\widetilde{Y}(x_k, T) = Y(x_k, T) = Y_k$ whose resolution requires in general $\mathcal{O}(N^2)$ operations.

However, we can choose adequate frequencies ω_j and evaluation points x_k so as to reduce the resolution cost to $\mathcal{O}(N \ln N)$ operations. In particular, we take uniformly spaced points[7]: $x_k = x_0 + k\Delta x$ and frequencies: $\omega_j = j\Delta\omega$ with: $\Delta\omega = 2\pi/N\Delta x$.

Under this parametrization, the N fitting conditions can be rewritten as[8]: $Y_k = \sum_{j=0}^{M-1} \psi_j e^{i\omega_j x_k} + \sum_{j=1}^{M-1} \left[\bar{\psi}_j e^{-i\theta_0}\right] e^{i\omega_{N-j} x_k}$ with: $\theta_0 = N x_0 \Delta\omega$.

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[4]For instance the payoff of a European vanilla call option with strike K: $V(x) = \max[e^x - K, 0]$ diverges to infinity as $x \to +\infty$ and hence is not square-integrable.

[5]The frequencies ω and the coefficients α and β are real. The constant term corresponds to the frequency: $\omega_0 = 0$. The coefficient β_0 is irrelevant as: $\sin\omega_0 x \equiv 0$ and is taken arbitrarily as: $\beta_0 = 0$ so that: $\psi_0 = \bar{\psi}_0 \in \mathbb{R}$.

[6]Based on the symmetries of the CF and payoff proxy, the pricing formula can be also written: $Y(X_0, 0) = \psi_0 + 2\sum_{j=1}^{M-1} Re\left[\psi^B_{X|X_0}(\omega_j, T)\psi_j\right]$

[7]For the sake of clarity, we define the space grid later -i.e. the first point x_0 and the spacing Δx.

[8]NB: $e^{-i\omega_j x_k} = e^{i\omega_{N-j} x_k} e^{-iNx_0\Delta\omega}$

Eventually, after replacing the points x_k and frequencies ω_j by their expressions, the coefficients of the payoff proxy $\widetilde{Y}$ satisfy the system:

$$Y_k = \sum_{j=0}^{N-1} \left[\tilde{\psi}_j e^{ij\Delta\omega x_0} \right] e^{2\pi ijk/N}$$

where:

$$\tilde{\psi}_j = \begin{cases} \psi_j & \text{if } j \in [0, M-1] \\ \bar{\psi}_{N-j} e^{-i\theta_0} & \text{if } j \in [M, N-1] \end{cases}$$

The Discrete Fourier Transform (DFT) transforms a vector Z of length N into another vector z of length N via the formula[9]: $z_k = \sum_{j=0}^{N-1} Z_j e^{\epsilon 2\pi ijk/N}$ where: $\epsilon = \pm 1$. It can be viewed as a system of N linear equations with variable Z. Remarkably, the solution is given by the inverse transform: $Z_j = 1/N \sum_{k=0}^{N-1} z_k e^{-\epsilon 2\pi ijk/N}$, which can be computed in $\mathcal{O}(N \ln N)$ operations with the Fast Fourier Transform (FFT) algorithm.

It follows that the coefficients ψ of the payoff proxy finally read:

$$\psi_j = \frac{1}{N} e^{-ij\Delta\omega x_0} \Phi_j$$

where: $\Phi = DFT\,(Y \mid \epsilon = -1)$ is the forward DFT Φ of the payoff vector Y of length N.

3.3 Gibbs's phenomenon and Space Grid

The numerical implementation of a Fourier transform sets lower and upper limits to the considered frequencies. This truncation acts like a band-pass filter in the frequency domain, which translates into a convolution with the cardinal sine[10] in the time domain. The resulting oscillations are the Gibbs's phenomenon and jeopardize the accuracy of the previous trigonometric proxy $\widetilde{Y}$.

Fortunately, we can specify the space grid -i.e. the first grid point x_0 and the spacing Δx- so as to address the Gibb's phenomenon as seen later in section 3.4. We assume that the probability mass of the state variable x lies between x_{min} and x_{max} given the initial value X_0.

In order to illustrate the Gibbs's phenomenon, we first set the space grid as: $x_0 = x_{min}$ and: $\Delta x = (x_{max} - x_{min}) / (N-1)$ and try to approximate the payoff of a European vanilla call option : $Y_k = \max\left[e^{x_k} - K, 0\right]$. The previous method enforces a perfect fit on the points x_k. However, the sinusoidal proxy $\widetilde{Y}(x, T)$ is not smooth between the points x_k as one might expect but oscillates as seen in figures 1 and 2. The oscillations condensate

[9] The DFT is called forward DFT for $\epsilon = -1$ and backward DFT for $\epsilon = +1$.

[10] The cardinal sine function is: $\text{sinc}\ \theta = \sin\theta/\theta$.

in the neighborhood of x_{min} and x_{max} as the resolution N of the space grid increases and their amplitude converges to a fixed value Λ.

The sinusoidal proxy $\widetilde{Y}$ has a period of $N\Delta x$ and therefore exhibits a sharp transition: $\Delta\widetilde{Y} = Y_{N-1} - Y_0$ at x^-_{min} and x^+_{max}. It has been shown [2] that the oscillations Λ are proportional[11] to the discontinuity $\Delta\widetilde{Y}$.

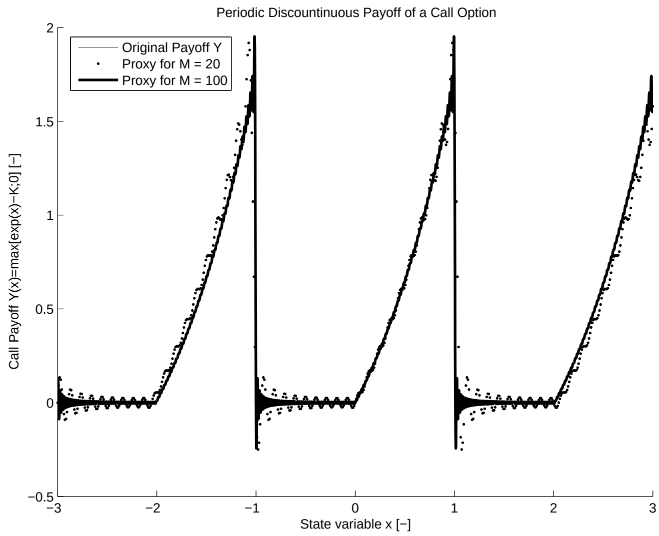

Figure 1: The state variable x ranges from -1 to $+1$ but the proxy is plotted between -3 and $+3$. The payoff of a call option (thin continuous line) reads $Y(x) = \max[e^x - K, 0]$ with strike $K = 1$. The sinusoidal proxy $\widetilde{Y}$ is periodic and has a discontinuity: $\Delta\widetilde{Y} = e - 1 \simeq 1.718282$ at $x = \pm 1$.

3.4 Symmetric payoff proxy

In order to alleviate the Gibb's phenomenon, we eliminate the discontinuity at x_{min} and x_{max} by altering the payoff Y. To do so, we set the space grid as[12]: $x_0 = x_{min}$ and: $\Delta x = 2(x_{max} - x_{min})/N$ and symmetrize the payoff as: $Y_{M+k} = Y_{M-1-k}$ for: $k \in [0, M-2]$. This alteration will impact the option price marginally because it occurs in an area of low probability of the state vector x.

This symmetrization dramatically improves the accuracy of the proxy as seen in figures 3 and 4. Also, the number of frequencies M plays a minor role as the oscillations quickly disappear for $M > 20$.

[11] The asymptotic amplitude of the oscillations Λ at x_{max} relates to the amplitude $\Delta\widetilde{Y}$ of the discontinuity $\Delta\widetilde{Y}$ as: $\Lambda = \varrho\Delta\widetilde{Y}$, where: $\varrho = 1/\pi \int_0^\pi \text{sinc}\ \theta d\theta - 1/2 \simeq 0.089409$. The number $\int_0^\pi \text{sinc}\ \theta d\theta$ is the Wilbraham-Gibbs constant.

[12] Note that the term Y_0 appears only once in the symmetrization.

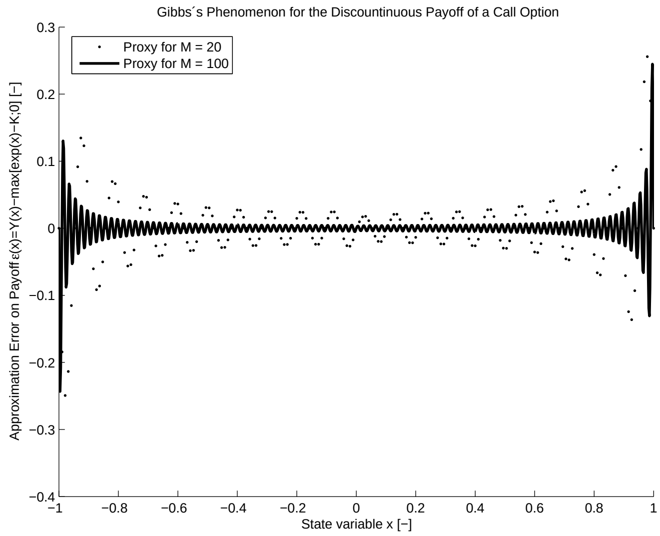

Figure 2: The state variable x ranges from -1 to $+1$. The payoff of a call option (thin continuous line) reads $Y(x) = \max\left[e^x - K, 0\right]$ with strike $K = 1$. Its sinusoidal proxy $\widetilde{Y}$ matches the original payoff Y at the points x_k but exhibits oscillations which are largest at $x = \pm 1$. As the number of frequencies M increases from 20 (dotted line) to 100 (thick continuous line), the oscillations vanish for $-1 < x < 1$ but converge in amplitude to $\Lambda \simeq 1.718282 \times 0.089409 \simeq 0.153630$ at $x = \pm 1$.

3.5 Summary

In summary, in order to price a European option with a continuous payoff[13] it is critical to find a good proxy to mitigate the Gibbs's oscillations. We do this by symmetrizing the discounted payoff Y. More specifically:

State variable Determine the confidence interval of the state vector x: x_{min} and x_{max}.

Space grid Consider $M > 20$ points x_k as: $x_k = x_{min} + k\Delta x$ where:

$$\Delta x = 2\left(x_{max} - x_{min}\right) / \left(2M - 1\right)$$

Terminal discounted payoff Compute for each of the grid points x_k the value of the discounted payoff as:

$$Y_k = \frac{V\left(x_k, T\right)}{B\left(x_k, T\right)}$$

Frequencies Select $N = 2M - 1$ frequencies ω_j as: $\omega_j = j\Delta\omega$ where:

$$\Delta\omega = 2\pi / N\Delta x$$

[13] The described method does not work for discontinuous payoffs such as e.g. digitals.

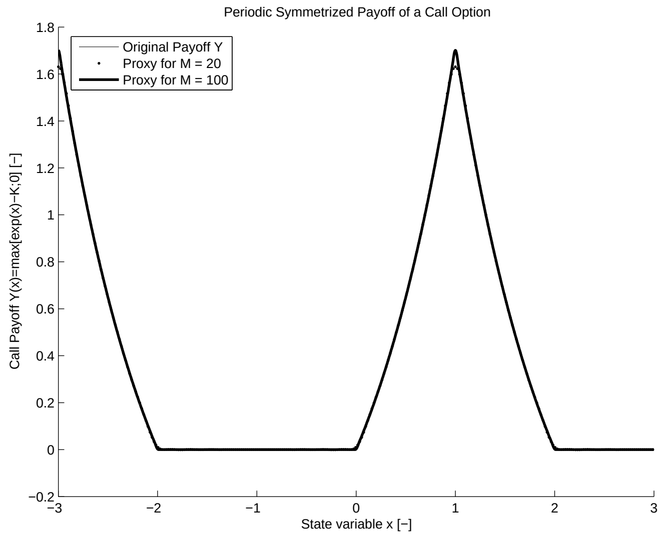

Figure 3: The settings are identical to figure 1. The original payoff Y (thin continuous line) is approximated by a symmetrized proxy $\widetilde{Y}$ with a number of frequencies M from 20 (dotted line) to 100 (thick continuous line).The symmetrized proxy is periodic and shows small oscillations.

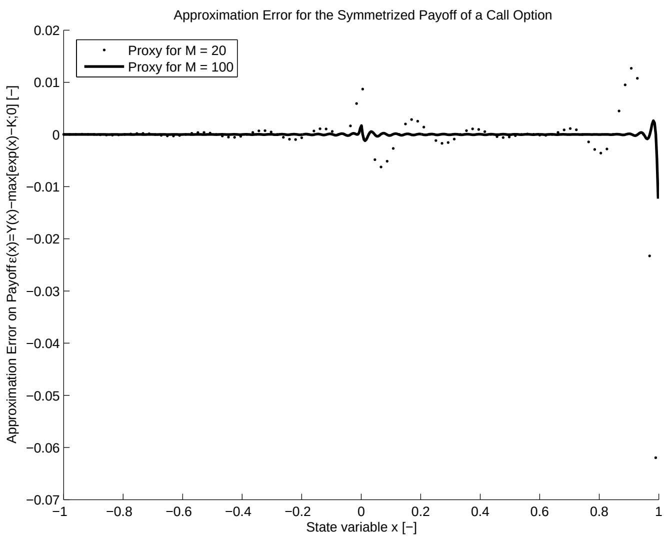

Figure 4: The settings are identical to figure 2. The original payoff Y (thin continuous line) is approximated by a symmetrized proxy $\widetilde{Y}$ with a number of frequencies M from 20 (dotted line) to 100 (thick continuous line). The Gibbs's phenomenon disappears even for $M = 20$. Note the scale difference with figure 2.

Symmetrized payoff Expand the discount payoff vector Y of length M into a vector of length N as:

$$Y_{M+k} = Y_{M-1-k} \text{ for: } k \in [0, M-2]$$

Proxy coefficients Compute the N coefficient of the payoff proxy $\widetilde{Y}$ as:

$$\psi_j = \frac{1}{N} e^{-ij\Delta\omega x_0} \Phi_j \text{ where: } \Phi = DFT\left(Y \mid \epsilon = -1\right)$$

Option price Obtain the price of the option at the point X_0 as:

$$V\left(X_0, 0\right) = B\left(X_0, 0\right) \left[\sum_{j=-(M-1)}^{M-1} \psi^B_{X|X_0}\left(\omega_j, T\right) \psi_j \right]$$

or:

$$V\left(X_0, 0\right) = B\left(X_0, 0\right) \left[\psi_0 + 2 \sum_{j=1}^{M-1} Re\left(\psi^B_{X|X_0}\left(\omega_j, T\right) \psi_j \right) \right]$$

4 Pricing Simultaneously for a Set of Initial Values

The previous pricing formula actually allows to discount an arbitrary European payoff for an initial point X_0 by simply taking the dot product of two Fourier transforms. This section explains how to efficiently apply this formula simultaneously on a set of initial values when the dynamics of the state variables are arithmetic. As a result, we obtain a price vector that allows to estimate the Greeks numerically.

4.1 Arithmetic dynamics and frequential fundamental relation

In this section, we consider only models where the innovation dx does not depend on the value x of the state vector, which implies that the CF $\psi^B_{X|X_0}$ isolates the contribution of the initial value X_0 as:

$$\psi^B_{X|X_0}\left(\omega, T\right) = e^{i\omega X_0} \psi^B_X\left(\omega, T\right)$$

Plugging this expression into the pricing formula and recognizing the inverse Fourier transform $\psi_Y\left(\omega, 0\right)$ of the price $Y\left(x, 0\right)$ leads to the frequential version of the fundamental pricing formula[14]:

$$\psi_Y\left(\omega, 0\right) = \psi^B_X\left(\omega, T\right) \psi_Y\left(\omega, T\right)$$

Apart from the benefits highlighted in section 2, this formula does not dependent the state variable x. Thus, a properly implemented DFT will compute the market price $Y\left(x, 0\right)$ for an entire set of state variables x_k.

[14] In the special case of a trigonometric payoff $\widetilde{Y}\left(x, T\right)$ as in section 3.1 leads to: $\widetilde{Y}\left(x, 0\right) = \sum_{j=-(M-1)}^{M-1} \left[\psi^B_X\left(\omega_j, T\right) \psi_j\right] e^{i\omega_j x}$.

4.2 Space grid determination

The space grid of section 3.2 centered on the current initial value X_0 is unfit because the symmetrization of section 3.4 distorts[15] the payoff proxy $\widetilde{Y}$ at the edges x_{min} and x_{max}.

In order to improve the accuracy of the proxy $\widetilde{Y}$ from x_{min} to x_{max}, we extend the space grid with the same space step $\Delta x = 2\left(x_{max} - x_{min}\right) / \left(2M - 1\right)$ from $x_0 = x_{min} - M\Delta x/2$ to $x_{2M-1} = x_{max} + \left(M - 1\right)\Delta x/2$. The number of frequencies then doubles to $M' = 2M$. Apart from these mild modifications, the whole procedure of section 3 remains valid with $N' = 2M' - 1$. Figure 5 offers a schematic comparison with the point-wise method of section 3.

4.3 Efficient discounting

Under the new space grid and frequency settings, redoing the algebraic manipulations as in section 3.2 shows that the present value vector[16] Y_k is a backward DFT:

$$Y = DFT\left(\hat{\psi} \mid \epsilon = +1\right)$$

where:

$$\hat{\psi}_j = \tilde{\psi}_j e^{ij\Delta\omega x_0}$$

and $\tilde{\psi}$ has the same definition as in 3.2 but replacing the coefficients ψ_j by $\psi_X^B\left(\omega_j, T\right)\psi_j$.

Finally, the option price at the M meaningful coordinates k from $\lceil M/2 \rceil$ to $\lfloor 3M/2 \rfloor$ reads:

$$V\left(x_k, 0\right) = B\left(x_k, 0\right) Y\left(x_k, 0\right)$$

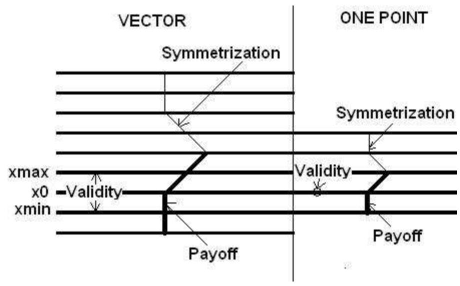

Figure 5: Comparison schema for the space grids and payoff symmetrizations for the point-wise method described in section 3 (right) and the vectorial method of section 4 (left).

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[15]As seen in figure 3 the proxy $\widetilde{Y}$ for a European vanilla call option is triangular at x_{max} whereas in fact it should be close to a straight line.

[16]The price vector Y has a length $Q = 2P - 1$

5 Numerical Example

We now illustrate the aforementioned results with the Black-Scholes model from [3] that assumes that the instantaneous returns of a non-dividend paying stock S are normally distributed with volatility σ. The numeraire B is a risk-free Money-Market Account continuously compounded at rate r: $B(T) = e^{rT}$. In order to obtain arithmetic dynamics, we choose the state variable $x = \ln S$. The CF ψ_X^B then becomes: $\psi_X^B(\omega, T) = e^{-\sigma^2 T\left(i\omega+\omega^2\right)/2}$ as seen in [1].

We look at the market prices V of an At-The-Money[17] (ATM) European vanilla call option computed successively for each point as in section 3 and simultaneously for all initial values as in section 4 shown in figure 6. A numerical differentiation allows to estimate the Greeks.

The vectorial method of section 4 generally yields a better accuracy for the price and its Greeks than the point-wise method of section 3 as seen in figures 7, 8, and 9. Moreover, the computation time of the vectorial method seems several orders of magnitude lower than its point-wise counterpart applied successively N times as seen in figure 10.

We finally repeat this study for an ATM European binary call option with strike $K = 100$ and maturity $T = 1$ year. Figure 11 suggests that the results are still accurate despite its payoff discontinuity at $x = \ln K$.

6 Conclusion

The fundamental Fourier-based pricing formula for an initial condition X_0 of section 2 is an integral and as such does not require costly resolutions of linear equations. Its inputs are the CF $\psi_{X|X_0}^B$ of the model and the Fourier transform ψ_Y of the terminal payoff.

Unfortunately, many contracts -e.g. European vanilla calls- do not admit such a transformation. Also, the Gibb's phenomenon plagues its numerical implementation with oscillations when the payoff profile is not periodic. Section 3 overcomes this difficulty by first setting an appropriate space grid and frequencies that remarkably depend only on the model and then designing an adequate symmetrized sinusoidal proxy for arbitrary payoff profiles. As a fortunate by-product, the previous integral becomes a finite sum. Also, there is no need to maintain a catalog of formulae for the Fourier transforms of payoffs.

Moreover, it is possible to efficiently discount the payoff simultaneously on a vector of initial conditions with a FFT when the model dynamics are arithmetic as seen in section 4 by extending the space grid.

[17] The space grid limits x_{min} and x_{max} need to be adapted for deep Out-The-Money or In-The-Money options.

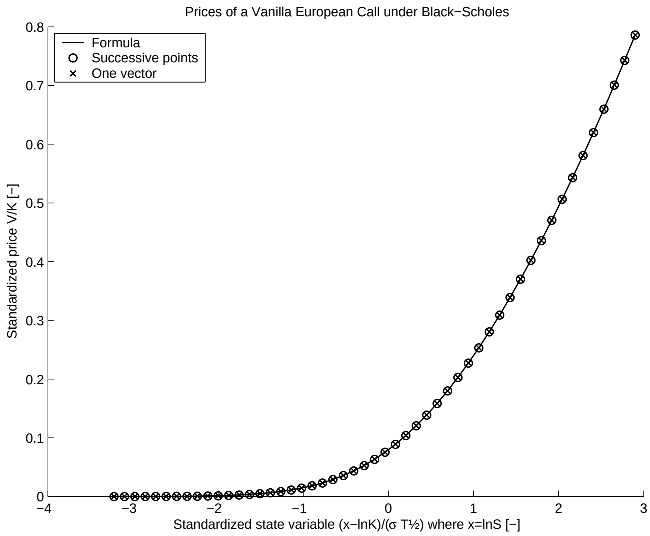

Figure 6: The settings are identical to figure 1 with $M = 50$ grid points between x_{min} and x_{max}. The price of a European vanilla call with strike $K = 100$ and maturity $T = 1$ is computed under the Black-Scholes model with volatility $\sigma = 0.20$, $M = 50$ frequencies, and CF $\psi_X^B(\omega, T) = e^{-\sigma^2 T\left(i\omega+\omega^2\right)/2}$. The initial underlying value is $S_0 = 100$ and the discount rate is $r = 0$. The state variable $x = \ln S$ is taken between $x_{max/min} = \ln S_0 - rT - \sigma^2 T/2 \pm 3\sigma\sqrt{T}$. The vectorial method (crosses) of section 3 yields directly the entire price vector with two FFT whereas the point-wise method (circles) of section 4 must be applied successively with one FFT on each of the M grid points. The true value (plain line) is given by the Black-Scholes formula.

The numerical example of section 5 suggests that the presented methodologies allows to compute accurate market prices and estimate high-quality Greeks through numerical differentiation.

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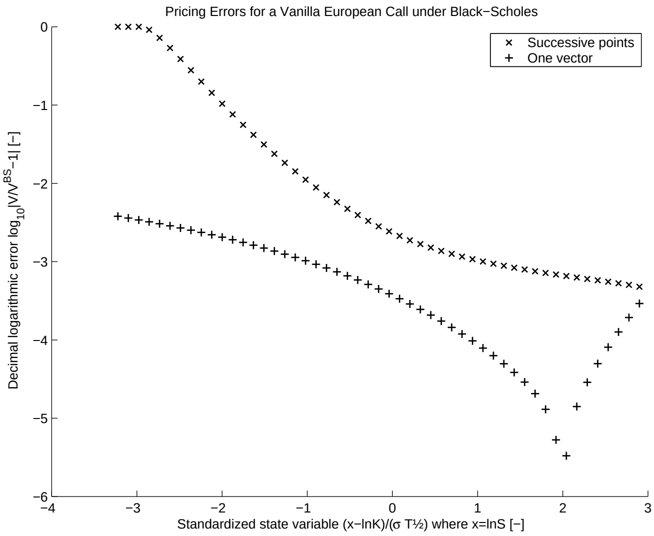

Figure 7: The settings are identical to figure 6. The graph compares the decimal logarithm of the relative error $\epsilon = \log_{10} \mid V/V^{BS} - 1 \mid$ of the vectorial method (crosses) of section 3 against the point-wise method (crosses) of section 4.

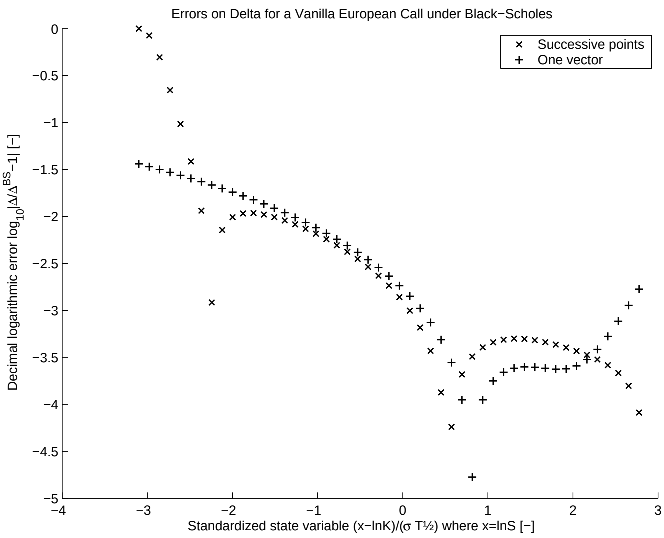

Figure 8: The settings are identical to figure 6. The graph shows the decimal logarithm of the relative error $\epsilon = \log_{10} \mid \Delta/\Delta^{BS} - 1 \mid$ for the first-order Greek $\Delta = \partial V/\partial S_0$. It compares the vectorial method (pluses) of section 3 with the point-wise method (crosses) of section 4.

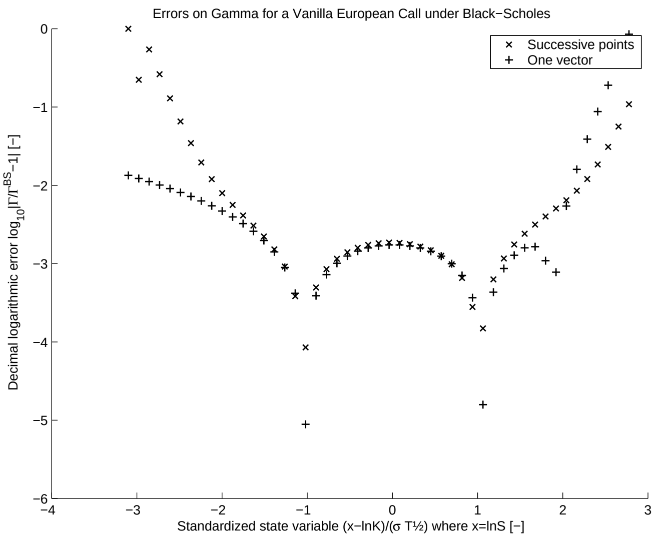

Figure 9: The settings are identical to figure 6. The graph shows the decimal logarithm of the relative error $\epsilon = \log_{10} \mid \Gamma/\Gamma^{BS} - 1 \mid$ for the second-order Greek $\Gamma = \partial^2 V/\partial S_0^2$. It compares the vectorial method (pluses) of section 3 with the point-wise method (crosses) of section 4.

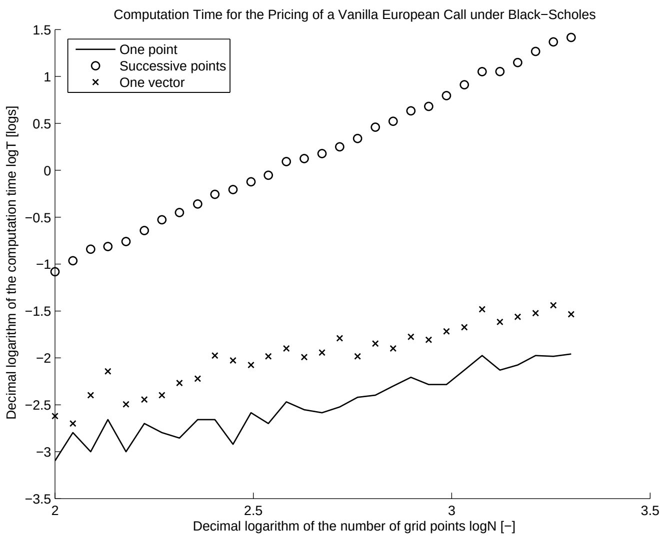

Figure 10: The settings are identical to figure 6. The graph plots the decimal logarithm of the computation time $\ln T$ as a function of the decimal logarithm of the number of grid points $\ln N$ when the pricing is performed for one single point (plain line) or N successive points (circles) as in section 3 or for a vector of N points (crosses) as in section 4.

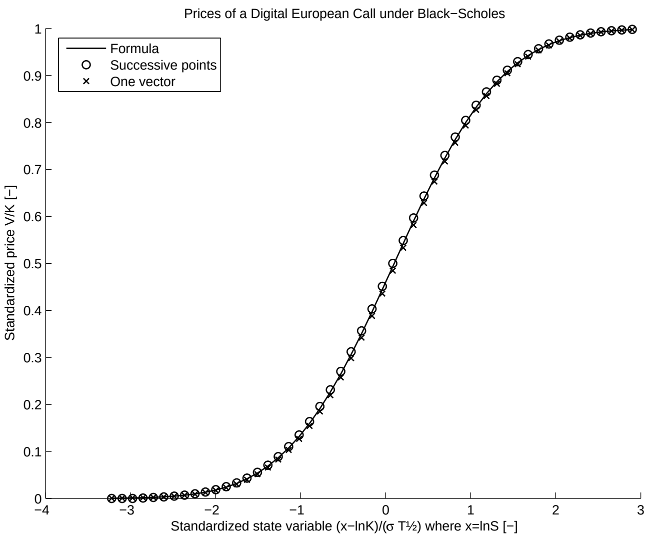

Figure 11: The settings are identical to figure 7, except that the payoff is binary. The vectorial (pluses) and the pointwise methods (crosses) still lead to an acceptable accuracy.