Modeling a Balance Sheet's Liabilities

Jacques Burrus

Burrus Financial Intelligence, Las Condes, Chile
E-Mail: Jacques.Burrus@bfi.cl

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A mi Polillita.

1 Introduction

Corporations finance their activities by emitting equity and bonds. Both classes of liabilities are closely related. For instance, an adverse piece of news typically results in a fall in the value of equity and an increase of the bond spreads.

This paper presents a model for the dynamics of equity and bonds with two state variables. First, the dividend level represents the profitability and follows a Geometric Lévy process. Then, the time to default measures the likelyhood of bankruptcy across time and undergoes a Feller diffusion. It is similar to a continuous credit rating and relates the evolution of equity with that of bond spreads.

The calibration requires only market data so that the state variables reflect the market consensus, not a particular fundamental analysis.

Closed formulas are available for the value of equity and its European vanilla derivatives as well as for the corporate pure discount bond and credit default swaps.

2 Dividend Level and Time to Default

The model incorporates two state variables that are the dividend level D and the time to default T.

2.1 Dividend Level

The dividend level D reflects the anticipated long-term profits and hereby dividend payments. It disregards the firm's leverage ratio. Daily news influence heavily the market profit anticipations and hence the dividend level D is highly volatile.

Under the physical measure P, the dividend level is assumed to follow a Geometric Lévy process:

$$D_t = D_0 e^{\mu^P t + \sigma Z_t^P}$$

Z_t is a standard Lévy process with defined Cumulant Generating Function[1] (CGF) $t\phi_Z$ and scaling coefficient σ. In the example of the Black-Scholes framework, Z_t is a Wiener process with instantaneous CGF $\phi_Z(\nu) = -\nu^2/2$.

The firm pays the dividends at the deterministic dates τ_m, e.g. quarterly. The future value of the dividend $\gamma_t(\tau_m)$ paid at date τ_m in the absence of bankruptcy conditional on a date t and the corresponding dividend level D_t reads:

$$\gamma_t(\tau_m) = \alpha_m \frac{D_t}{P_t(\tau_m)}$$

The domestic risk-free discount factor $P_t(\tau)$ accounts for the time-value of money. The factors α are by default all equal to 1. However, an economist may modulate the dividends across time. For instance, a company would pay out lower dividends in its growth phase as it needs to fund its expansion. The dividends later increase during the maturity phase before eventually fading in the dedine phase.

2.2 Time to Default

The time to default T measures the time left until bankruptcy. The larger its value, the more perennial the firm. In particular, bankupcy is characterized by a time to default of zero. Here, the time to default T follows a continuous branching process also known as Feller diffusion.

2.3 Discrete branching process

Branching processes model e.g. the spread of surnames in Genealogy, the evolution of a specie's population in Biology, or the propagation of genes in Genetics. A population's extinction -like bankruptcy- under a branching process is irreversible. In this model, a population increase -i.e a larger time to default T-corresponds to a properous period.

Galton-Watson [1] is the simplest branching process and assumes discrete time periods n and population sizes X_n. Each individual i gives birth to Y_i offsprings and dies at the end of the period so that the population sizes between two consecutive generations relate as: $X_{n+1} = \sum_{i=1}^{X_n} Y_i$. Clearly,

[1]The Cumulant Generating Function is the natural logarithm of the Moment Generating Function (MGF). By definition, the MGF ψ of a random variable X is: $\psi(\nu) = E\left[e^{\nu X}\right]$

extinction is an absorbing state. The number of offsprings is independent and identically distributed across individuals with CGF ϕ_Y. Their respective CGF's therefore verify[2]: $\phi_{X_{n+1}} = \phi_{X_n} \circ \phi_Y$.

2.4 Feller diffusion

By assumption, the time to default T is a continuous quantity continuously updated by the markets. It thus follows a Feller diffusion[2] which is the continuous limit of the aforementioned Galton-Watson process. Let Δt denote the duration of an infinitesimal time period. The incremental number of offsprings $\Delta Y = Y - 1$ is then infinitesimal with -per hypothesis- a normal distribution[3] with growth rate κ and volatility η. Its CGF hence reads: $\phi_{\Delta Y}(\nu) = \left(\kappa\nu + \eta^2\nu^2/2\right)\Delta t$. Eventually, the Galton-Watson relation for the CGF ϕ_T becomes: $\phi_T(\nu, t + \Delta t) = \phi_T\left(\nu + \left(\kappa\nu + \eta^2\nu^2/2\right)\Delta t, t\right)$. A Taylor expansion shows that the CGF obeys the partial differential equation:

$$\frac{\partial \phi_T}{\partial t} = \left(\kappa\nu + \eta^2\nu^2/2\right)\frac{\partial \phi_T}{\partial \nu}$$

with initial condition:

$$\phi_T(\nu, 0) = T_0\nu$$

The CGF of the time to default T [4] finally has the analytic solution [3]:

$$\phi_T(\nu, t) = T_0\nu e^{\kappa t}\left[1 + \frac{\eta^2\nu}{2\kappa}\left(1 - e^{\kappa t}\right)\right]^{-1}$$

2.5 Survival probability

The survival probability $\Lambda_0(\tau)$ of the firm until a future date τ -i.e. the probability of no bankruptcy- is by definition the probability that the future time to default T_τ is strictly positive: $\Lambda_0(\tau) = P[T_\tau > 0 \mid T_0]$. It takes the analytical form[2]:

$$\Lambda_0(\tau) = 1 - e^{\frac{-2\kappa T_0}{\eta^2(1-e^{-\kappa\tau})}}$$

It depends on an adimensional time to default: $Q_0 = 2\kappa T_0/\eta^2$.

The survival probability term structure $\Lambda_0(\tau)$ exhibits two flat regions at $\tau = 0$ and $\tau \to +\infty$. Notably on the long run, the firm has a positive survival

[2]The convolution theorem states that the CGF of two independent random variables is the sum of their CGF's. Hence, the CGF of X_{n+1} conditional on X_n reads: $\phi_{X_{n+1}|X_n} = X_n\phi_Y$. The unconditional CGF follows as: $\phi_{X_{n+1}} = \ln E\left[e^{\phi_{X_{n+1}|X_n}}\right]$.

[3]The incremental number of offsprings ΔY is a continuous quantity with an infinitely divisible distribution. The normal distribution satisfies these conditions and posseses a defined CGF.

[4]The probability density π_T of the time to default T at a future date t is: $\pi_T(x \mid t, T_0) = 2\kappa/\eta^2\left(e^{\kappa t} - 1\right)\sqrt{T_0 e^{\kappa t}/x}\, e^{-2\kappa\left(T_0 e^{\kappa t} + x\right)/\eta^2\left(e^{\kappa t} - 1\right)} I_1\left(4\kappa\sqrt{T_0 e^{\kappa t} x}/\eta^2\left(e^{\kappa t} - 1\right)\right)$ with the first order modified Bessel function of the first kind: $I_1(y) = \sum_{k=0}^{+\infty}\frac{(y/2)^{2k+1}}{k!(k+1)!}$.

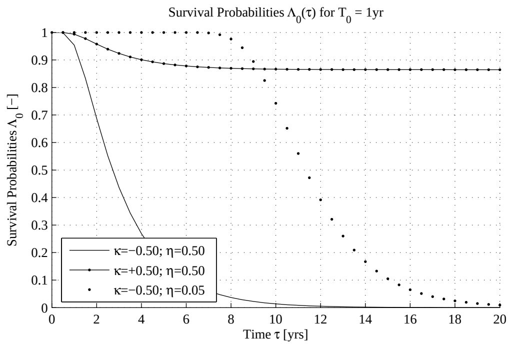

Figure 1: The survival probability $\Lambda_0(\tau)$ term structure is flat in the short and long run. The inflexion point $\tau^\star$ separates both regions.

probability $\Lambda_0(+\infty) = 1 - e^{-Q_0}$ when the growth rate κ is strictly positive. Otherwise it almost surely fails, i.e. $\Lambda_0(+\infty) = 0$. The transition occurs at the inflexion point: $\tau^\star = \ln\left[\frac{\sqrt{Q_0^2+4}+Q_0}{2}\right]/\kappa$ where the firm's situation most dramatically deteriorates. The precision $\Delta\tau^\star$ of this critical date is the time scale of the transition[5]: $\Delta\tau^\star = \frac{2[\Lambda(0)-\Lambda(+\infty)]}{\kappa Q_0}\sinh^2(\kappa\tau^\star/2)\, e^{Q_0/\left(1-e^{-\kappa\tau^\star}\right)}$.

3 Equity and Corporate Bonds

In order to preserve internal consistency (put-call parity) and later price derivatives on the firm's liabilities, one must convert the dynamics of the dividend level D and time to default T into a martingale equivalent measure. This article chooses the risk-neutral measure Q, i.e. the numeraire is a risk-free money market account (MMA) B compounded by the the possibly stochastic risk-free short rate r: $B_t = e^{\int_{\tau=0}^{t} r_\tau d\tau}$. The process of the risk-free curve is assumed independent from the dividend level D and time to default T.

3.1 Equity

Under the risk-neutral measure Q, the market stock price S must satisfy the martingale relation:

$$\frac{S_0}{B_0} = \sum_{\tau_m<t} E_0^Q\left[\frac{\gamma_{\tau_m}(\tau_m)}{B_{\tau_m}}\right] + E_0^Q\left[\frac{S_t}{B_t}\right]$$

The switch from the physical measure P to the risk-neutral measure Q

[5]The time scale of the transition $\Delta\tau^\star$ satisfies by definition at the inflexion point $\tau^\star$: $[\Lambda(+\infty) - \Lambda(0)]/2\Delta\tau^\star = \partial\Lambda/\partial\tau$. It also measures the sharpness of the transition.

requires to alter the drift of the dividend level D as: $D_t = \frac{D_0}{P_0(t)} e^{-\phi_Z(\sigma)t+\sigma Z_t^Q}$, while the dynamics of the time to default T remain unchanged.

The observations that: $E_0^Q [D_t/B_t] = D_0/B_0$ and: $E_0^Q [\Lambda_t (\tau)] = 1 - \psi_T \left(-\kappa/\eta^2 (1 - e^{-\kappa\tau}), t\right) = \Lambda_0 (\tau)$ lead to the aforementionned martingale relation.

The market price of equity S then turns out to be the present value of the dividend stream:

$$S_0 = \sum_{\tau_m \geq 0} \gamma_0 (\tau_m) P_0 (\tau_m) \Lambda_0 (\tau_m)$$

and is in fact proportional to the dividend level D:

$$S_0 = D_0 \left(\sum_{\tau_m \geq 0} \alpha_m \Lambda_0 (\tau_m) \right)$$

The payout ratio $\zeta (T_0) = S_0/D_0$ depends positively on the sole time to default T_0 and falls exactly to zero in the event of bankruptcy: $T_0 = 0$.

3.2 Corporate Bonds

The Pure Discount Bonds (PDB) of the corporation are the building block for the valuation of more elaborate bonds. Let $P_t^{CORP} (\tau|\rho)$ denote the market price at date t of a PDB with tenor τ and seniority ρ. The seniority ρ is the recovery rate in case of default: 1 for most senior; 0 for the most junior debt. At expiration, this PDB pays in the absence of default before τ one unit currency and ρ otherwise.

The market price of a PDB reads[6]:

$$\frac{P_0^{CORP} (\tau|\rho)}{B_0} = E_0^Q \left[\frac{1}{B_\tau} \right]$$

which eventually becomes:

$$P_0^{CORP} (\tau|\rho) = P_0 (\tau) [\rho + (1 - \rho) \Lambda_0 (\tau)]$$

The corporate spread $\xi (T_0|\tau, \rho) = P_0^{CORP} (\tau|\rho) / P_0 (\tau)$ depends only on the time to default T and increases as the time to default T decreases.

3.3 Capital Structure Arbitrage

The time to default T appears in both the payout ratio $\zeta (T)$ and the corporate spread $\xi (T|\tau, \rho)$ and is therefore responsible for negatively relating the prices of equity S and bonds P^{CORP}.

[6] The market price of the PDB clearly satisfies the martingale relation: $P_0^{CORP} (\tau|\rho)/B_0 = E_0^Q \left[P_t^{CORP} (\tau|\rho)/B_t\right]$

For example, all else being equal, a prosperous business period leads to a higher time to default T. The survival probabilities Λ then increase yielding both higher payout ratios ζ and lower spreads ξ.

Capital structure arbitrage may rely on this explicit relation between the various corporate liabilities.

4 Derivatives

The model offers closed formulae for a variety of derivatives on corporate liabilities.

4.1 European Vanilla Stock Options

The market price V_0 of a European vanilla options on the corporation's equity S with maturity t and strike K can be written in terms of the integral:

$$V_0 = \int_{x=0}^{+\infty} BSL\left[K, t \mid D_0 \zeta(x), P_0(t), \sigma\right] \pi_T(x \mid t, T_0)\, dx$$

where BSL[7] refers to the Black-Scholes-Lévy formula[4] for a European vanilla option.

4.2 Credit Default Swaps

A Credit Default Swap (CDS) insures a PBD over N periods extending from τ_0 to τ_N. The protection buyer pays a premium p at the end of each period until default, in which case he delivers the failed PBD -worth the recovery rate ρ in [5] - to the protection seller and receives the face value.

The price V_0 of the CDS then reads[8]:

$$V_0 = \left[\sum_{i=1}^{N} p\Lambda_0(\tau_i) P_0(\tau_i)\right] - \left[\sum_{i=1}^{N} (1-\rho)\left[\Lambda_0(\tau_{i-1}) - \Lambda_0(\tau_i)\right] P_0(\tau_i)\right]$$

and depends only on the adimensional time to default Q_0.

[7] The function BSL is the usual Black-Scholes formula when Z undergoes a Wiener process, i.e. $\phi_Z(\nu) = -\nu^2/2$.

[8] The market usually quotes the premium $p_0^\star$ for which the CDS is free: $p_0^\star = (1-\rho)\left[\sum_{i=1}^{N}\left[\Lambda_0(\tau_{i-1}) - \Lambda_0(\tau_i)\right] P_0(\tau_i)\right] / \left[\sum_{i=1}^{N} \Delta\tau_i \Lambda_0(\tau_i) P_0(\tau_i))\right]$.

4.3 Convertible Bonds

5 Example of Citi

5.1 Data

Citi experienced a turning point in June 2007 shown in figure 2. Before that, its equity was oscillating around USD 50/share and CDS premia were about 30bp/year. By March 2009, shares fell to about USD 1/share while CDS premia surged to 500bp/year. The evolution of equity and credit henceforth looks indeed strongly related.

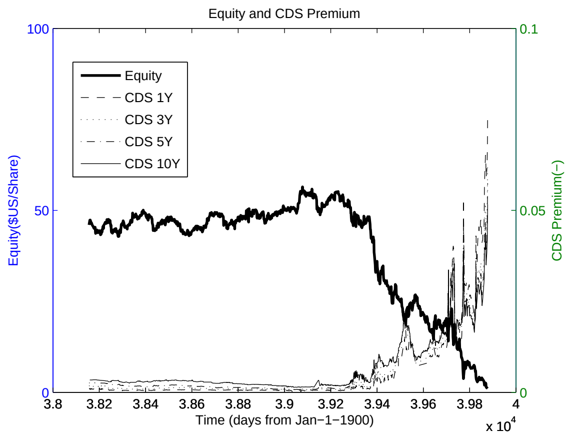

Figure 2: Citi's equity (thick solid line) and CDS premium on senior bonds for 1Y (thin dashed line), 3Y (thin dotted line), 5Y (thin dashed-dotted line), and 10Y (thin solid line) from June 17th 2004 to March 10th 2009.

5.2 Calibration of the Time to Default with CDS

5.2.1 Methodology

As seen in section 4.2, the calibration using CDS premia will allow to extract the adimensional time to default Q, the growth rate κ, and the recovery rate ρ.

First - conditional on a recovery rate ρ - we bootstrap the empirical survival probabilities $\widehat{\Lambda}_{ij}$ for each day t_i and available tenors[9] τ_j using the CDS premia $p^{\star}_{ij}$.

Then for each date t_j - conditional on a growth rate κ and the previous recovery rate ρ - we determine the adimensional time to default $Q_j = Q_{t_j}$ by

[9]There are 4 tenors (1Y, 3Y, 5Y, and 10Y) available per day in the data set of figure1.

minimizing[10] e.g. the standard deviation of the errors between the empirical and model survival probabilities: $|\Lambda_{t_j}(\tau_i) - \widehat{\Lambda}_{ij}|$. Let ϵ_j denote the minimum.

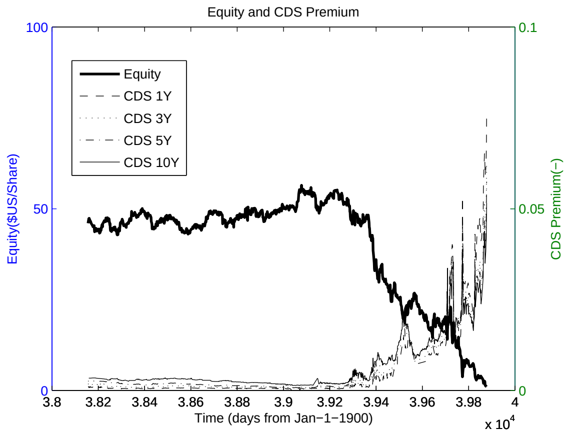

Figure 3: Citi's survival probability term structure $\Lambda_t(\tau)$ as of June 1st 2007 with a recovery rate $\rho = XXXX$, a growth factor $\kappa = YYY$, and adimensional time-to-default $Q_t = ZZZ$. The empirical bootstrapped probabilities (pluses) are fitted first by an OLS (crosses) and then by a minimization (circles).

Next - conditional on the recovery rate ρ - a minimization of e.g. the standard deviation of ϵ_j provides the growth rate κ. This minimum is $\epsilon^\star$ and depends only on the recovery rate ρ.

Finally, a minimization[11] of $\epsilon^\star(\rho)$ gives the recovery rate ρ.

5.2.2 Time to default and criticial time

5.2.3 Comparison with credit ratings

Credit rating agencies accompany their ratings (bankrupcy is the lowest grade) with a transition matrix that allows to back out[12] the survival probability term structure.

[10] Assuming that $|\kappa\tau| \ll 1$, a second order Taylor expansion on $\delta_{ij} = -\ln\left[1 - \widehat{\Lambda}_{ij}\right] = Q_j / \left(1 - e^{-\kappa\tau_i}\right)$ yields: $\delta_{ij} \approx Q_j\left(1/2 + 1/\kappa\tau_j\right) = \beta_0 + \beta_1\left(1/\tau_j\right)$. An Ordinary Least Squares (OLS) regression gives an estimate $\hat{\beta}$ for the intercept and the slope. Using the data for the first date t_0, an initial guess for the growth rate is: $\hat{\kappa} = 2\hat{\beta}_1/\hat{\beta}_0$, and for the adimensional time to default: $\widehat{Q}_0 = 2\hat{\beta}_0$. For a latter date t_j, an initial guess for the adimensional time to default is the adimensional time to default obtained from the minimization at t_{j-1}: $\widehat{Q}_j = Q_{j-1}$.

[11] This minimization requires to bootstrap the empirical survival probabilities $\widehat{\Lambda}$ at each iteration. The initial guess for the recovery rate ρ is not important because the objective

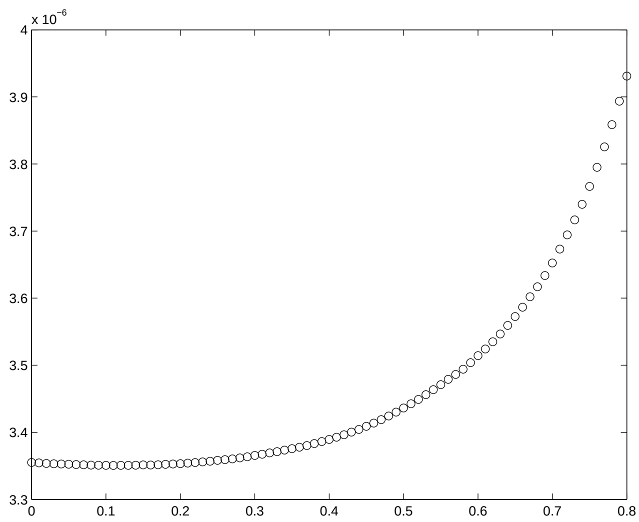

Figure 4: The global fitting error $\epsilon^{\star}$ seems to be a convex function of the recovery rate ρ.

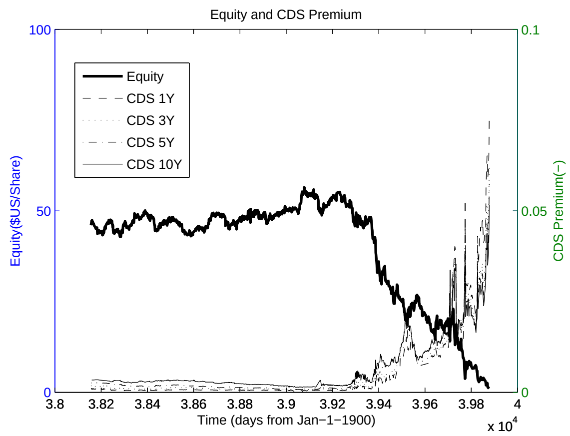

Figure 5: The calibration based on the CDS premia from June 17th 2004 to March 10th 2009 yields the adimensional time to default Q (dashed line) and the critical date $\tau^{\star}$ (thick solid line) and transition scope $\Delta\tau^{\star}$ (thin solid lines).

function $\epsilon^{\star}(\rho)$ looks convex in figure 4 and therefore admits only one global minimum.

[12]The Chapman-Kolmogorov equation provides the probability distribution of the ratings in time.

	All data	First period	Second period
Growth rate κ	1	0.96595	0.96595
Recovery rate ρ	1	2	3

Table 1: The calibration methodology leads to different growth rates κ and recovery factors ρ depending on the selected data: June 17th 2004 to March 10th 2009 (all data), June 17th 2004 to June 1st 2007 (first period), and June 1st 2007 to March 10th 2009 (second period), suggesting a regime switch around June 2007.

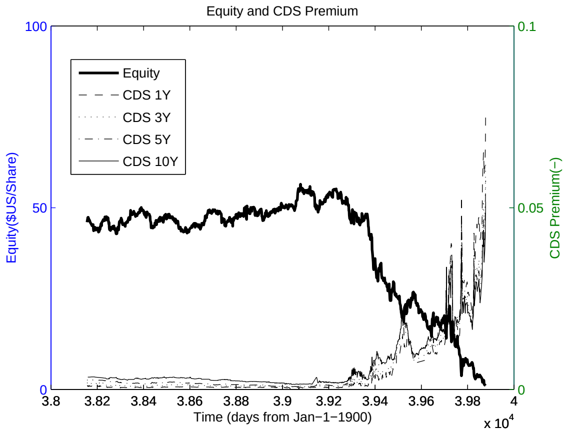

Figure 6: The survival probability at 1Y (pluses), 3Y (crosses), 5Y (diamonds), and 10Y (circles) is consistently higher for the credit rating agency Fitch (ordinate) than the model (abscissa) using the CDS premia from June 17th 2004 to March 10th 2009.

5.2.4 Survival probability extrapolation

5.3 Cross-sectional Calibration of the Volatility Surface

5.3.1 Methodology

5.3.2 Dividend level

5.3.3 Volatility skew

6 Conclusion

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